Integrales inmediates y sustituciones sencillas

1. $\int tg^3\,x\; sec^2 x\; dx$ Sea $u = tg\,x \Rightarrow du = sec^2\,x\,dx$

$$\int tg^3\,x\; sec^2 x\; dx = \int u^3\;du = \frac{u^4}{4} + C = \frac{tg^4\,x}{4} + C$$

2. $\int \frac{x^2}{\sqrt[4]{(x^3+1)^7}}\,dx$ Sea $u = x^3 + 1 \Rightarrow du = 3x^2\;dx$

$$\int \frac{x^2}{\sqrt[4]{(x^3+1)^7}}\,dx = \frac{1}{3}\int \frac{1}{u^{7/4}} = \frac{-4}{3\,u^{3/4}} + C = \frac{-4}{3\,(x^3+1)^{3/4}} + C$$

3. $\int \frac{sen\,\sqrt{x}}{\sqrt{x}}\,dx$ Sea $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx$

$$\int \frac{sen\,\sqrt{x}}{\sqrt{x}}\,dx = 2\int sen\,u\,du = -2\,cos\,u + C = -2\,cos\,\sqrt{x} + C$$

4. $\int \frac{2x-3}{4x^2-12x+1}\,dx = \frac{1}{4}\int \frac{8x-12}{4x^2-12x+1}\,dx$

Sea $u = 4x^2 - 12x + 1 \Rightarrow du = 8x - 12\;dx$

$$\int \frac{2x-3}{4x^2-12x+1}\,dx = \frac{1}{4}\int \frac{1}{u}\,du = ln\,u + C = ln\,|4x^2 - 12x + 1| + C$$

5. $\int \frac{Arcsen\,x}{\sqrt{1-x^2}}\,dx$ Sea $u = Arcsen\,x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}\,dx$

$$\int \frac{Arcsen\,x}{\sqrt{1-x^2}}\,dx = \int u\;du = \frac{u^2}{2} + C = \frac{Arcsen^2\,x}{2} + C$$

6. $\int x\;sen\,x^2\;cos\,x^2\;dx$ Sea $u = sen\,x^2 \Rightarrow du = 2x\,cos\,x^2\;dx$

$$\int x\;sen\,x^2\;cos\,x^2\;dx = \frac{1}{2}\int u\;du = \frac{u^2}{4} + C = \frac{sen^2\,x^2}{4} + C$$

7. $\int \frac{1}{x\sqrt{1-ln^2\,x}}\,dx$ Sea $u = ln\,x \Rightarrow du = \frac{1}{x}\,dx$

$$\int \frac{1}{x\sqrt{1-ln^2\,x}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = Arcsen\;u + C = Arcsen\,ln\,x + C$$

8. $\int \frac{1}{x\,\sqrt{x^2-1}\;Arcsec\,x}\,dx$ Sea $u = Arcsec\,x \Rightarrow du = \frac{1}{x\,\sqrt{x^2-1}}\,dx$

$$\int \frac{1}{x\,\sqrt{x^2-1}\;Arcsec\,x}\,dx = \int u\;du = \frac{u^2}{2} + C = \frac{Arcsec^2\,x}{2} + C$$

9. $\int \frac{ae^x + b}{ae^x - b}\,dx = \int \frac{ae^x + b}{ae^x - b}\,\frac{e^{-x/2}}{e^{-x/2}}\,dx = \int \frac{ae^{x/2} + be^{-x/2}}{ae^{x/2} - be^{-x/2}}\,dx$

Sea $u = ae^{x/2} - be^{-x/2} \Rightarrow du = \frac{1}{2}(ae^{x/2} + be^{-x/2})\, dx$

$$\int \frac{ae^x + b}{ae^x - b}\, dx = \frac{1}{2}\frac{1}{u}\, du = ln\, u + C = \frac{1}{2}\, ln\, |ae^{x/2} - be^{-x/2}| + C$$

10. Las siguientes integrales resultan inmediatas para cierto valor particular del número racional r. Determine en cada caso el valor de r y calcule la integral correspondiente.

a) $\int x^r\, e^{x^3}\, dx \qquad r = 2,\;\; u = x^3 \Rightarrow du = 3x^2\, dx$

$$\int x^2\, e^{x^3}\, dx = \frac{1}{3}\int e^u\, du = \frac{1}{3}\, e^u + C = \frac{1}{3}\, e^{x^3} + C$$

b) $\int x^r\, ln\, x\, dx \qquad r = -1,\;\; u = ln\, x \Rightarrow du = \frac{1}{x}\, dx$

$$\int x^{-1}\, ln\, x\, dx = \int \frac{ln\, x}{x}\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{ln^2\, x}{2} + C$$

c) $\int x^r\, sen\, \sqrt{x}\, dx \qquad r = -\frac{1}{2}\;\; u = \sqrt{x} \Rightarrow du = \frac{1}{\sqrt{x}\, dx}$

$$\int \frac{sen\, \sqrt{x}}{\sqrt{x}}\, dx = 2\int sen\, u\, du = -2cos\, u + C = -2cos\, \sqrt{x} + C$$

Integración por partes

11. $\int Arcsen \frac{2x}{1+x^2}\, dx$

Sea $u = Arcsen \frac{2x}{1+x^2} \;\wedge\; dv = \; dx \Rightarrow$

$$v = x \wedge du = \frac{1}{\sqrt{1 - \left(\frac{2x}{1+x^2}\right)^2}} \frac{2(1+x^2) - 2x * 2x}{(1+x^2)^2}\, dx = \frac{1}{\sqrt{\frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2}}} \frac{2 + 2x^2 - 4x^2}{(1+x^2)^2}\, dx =$$

$$\frac{1}{\sqrt{\frac{1+2x^2+x^4-4x^2}{(1+x^2)^2}}} \frac{2 - 2x^2}{(1+x^2)^2}\, dx = \frac{1+x^2}{\sqrt{1 - 2x^2 + x^4}} \frac{2 - 2x^2}{(1+x^2)^2}\, dx = \frac{1}{\sqrt{(1-x^2)^2}} \frac{2(1-x^2)}{(1+x^2)}\, dx =$$

$$\frac{2}{(1+x^2)}\, dx \Rightarrow du = \frac{2}{(1+x^2)}\, dx$$

Por partes queda:

$$\int Arcsen \frac{2x}{1+x^2}\, dx = x\, Arcsen \frac{2x}{1+x^2} - \int \frac{2x}{(1+x^2)}\, dx =$$

$$x\, Arcsen \frac{2x}{1+x^2} - ln\, (1+x^2) + C$$

12. $\int x\, Arctg\, (3x+4)\, dx$

Sea $u = Arctg\, (3x+4) \;\wedge\; dv = x\, dx \Rightarrow du = \frac{1}{1+(3x+4)^2}\, dx \;\wedge\; v = \frac{x^2}{2}$

Por partes queda:

$$\int x\, Arctg\,(3x+4)\; dx = \frac{x^2}{2}\, Arctg\,(3x+4) - \int \frac{x^2}{2}\, \frac{1}{1+(3x+4)^2}\; dx =$$

$\dfrac{x^2}{2}\, Arctg\,(3x+4) - \dfrac{1}{2}\displaystyle\int \frac{x^2}{9x^2+24x+17}\; dx$ Dividiendo los polinomios de la integral

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{1}{2}\int \frac{1}{9} + \frac{\frac{-24x-17}{9}}{9x^2+24x+17}\; dx$$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{1}{2}\int \frac{1}{9} - \frac{1}{2}\int \frac{\frac{-24x-17}{9}}{9x^2+24x+17}\; dx$$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{x}{18} + \frac{1}{18}\int \frac{24x+17}{9x^2+24x+17}\; dx$$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{x}{18} + \frac{18}{24}\int \frac{18x+\frac{51}{4}}{9x^2+24x+17}\; dx$$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{x}{18} + \frac{18}{24}\int \frac{18x+\frac{51}{4}+\frac{45}{4}-\frac{45}{4}}{9x^2+24x+17}\; dx$$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{x}{18} + \frac{18}{24}\int \frac{18x+24-\frac{45}{4}}{9x^2+24x+17}\; dx$$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{x}{18} + \frac{18}{24}\int \frac{18x+24}{9x^2+24x+17}\; dx - \frac{18}{24}\int \frac{\frac{45}{4}}{9x^2+24x+17}\; dx$$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{x}{18} + \frac{18}{24}\, ln\,|9x^2+24x+17| - \frac{18}{24}\,\frac{45}{4}\int \frac{1}{9x^2+24x+17}\; dx$$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{x}{18} + \frac{18}{24}\, ln\,|9x^2+24x+17| - \frac{135}{16}\int \frac{1}{(3x+4)^2+1}\; dx$$

Sea $t = 3x+4 \Rightarrow dt = 3\; dx$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{x}{18} + \frac{18}{24}\, ln\,|9x^2+24x+17| - \frac{135}{16}\,\frac{1}{3}\int \frac{1}{(t)^2+1}\; dt$$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{x}{18} + \frac{18}{24}\, ln\,|9x^2+24x+17| - \frac{45}{16}\, Arctg\, t + C$$

$$= \frac{x^2}{2}\, Arctg\,(3x+4) - \frac{x}{18} + \frac{18}{24}\, ln\,|9x^2+24x+17| - \frac{45}{16}\, Arctg\,(3x+4) + C$$

13. $\displaystyle\int x\, Arctg\,\sqrt{x^2-1}\; dx$

Sea $u = Arctg\,\sqrt{x^2-1} \;\wedge\; dv = x\; dx \Rightarrow du = \dfrac{1}{1+x^2-1}\,\dfrac{2x}{2\sqrt{x^2-1}}\; dx \;\wedge\; v = \dfrac{x^2}{2}$

$$\int x\, Arctg\,\sqrt{x^2-1}\; dx = \frac{x^2}{2}\, Arctg\,\sqrt{x^2-1} - \int \frac{x^2}{2}\,\frac{1}{x^2}\,\frac{2x}{2\sqrt{x^2-1}}\; dx$$

$$= \frac{x^2}{2}\, Arctg\,\sqrt{x^2-1} - \int \frac{x}{2\sqrt{x^2-1}}\; dx$$

Sea $t^2 = x^2 - 1 \Rightarrow 2t\, dt = 2x\, dx$

$$= \frac{x^2}{2}\, Arctg\, \sqrt{x^2-1} - \frac{1}{2}\int \frac{t}{\sqrt{t^2}}\, dt = \frac{x^2}{2}\, Arctg\, \sqrt{x^2-1} - \frac{1}{2}\int\, dt$$

$$= \frac{x^2}{2}\, Arctg\, \sqrt{x^2-1} - \frac{t}{2} + C = \frac{x^2}{2}\, Arctg\, \sqrt{x^2-1} - \frac{\sqrt{x^2-1}}{2} + C$$

14. $\displaystyle\int x^2\, e^{-x}\, dx$

Sea $u = x^2 \;\wedge\; dv = e^{-x}\, dx \Rightarrow du = 2x\, dx \;\wedge\; v = -e^{-x}$

$$\int x^2\, e^{-x}\, dx = -x^2 e^{-x} + \int 2x\, e^{-x}\, dx$$

Sea $u = 2x \;\wedge\; dv = e^{-x}\, dx \Rightarrow du = 2\, dx \;\wedge\; v = -e^{-x}$

$$\int x^2\, e^{-x}\, dx = -x^2\, e^{-x} - 2x\, e^{-x} + \int 2\, e^{-x}\, dx = -x^2\, e^{-x} - 2x\, e^{-x} - 2\, e^{-x} + C$$

15. $\displaystyle\int (Arcsen\, x)^2\, dx$

Sea $u = (Arcsen\, x)^2 \;\wedge\; dv = \; dx \Rightarrow du = 2\, Arcsen\, x\, \dfrac{1}{\sqrt{1-x^2}}\, dx \;\wedge\; v = x$

$$\int (Arcsen\, x)^2\, dx = x\,(Arcsen\, x)^2 - \int x\, 2\, Arcsen\, x\, \frac{1}{\sqrt{1-x^2}}$$

$$\int (Arcsen\, x)^2\, dx = x\,(Arcsen\, x)^2 - 2\int Arcsen\, x\, \frac{x}{\sqrt{1-x^2}}$$

Sea $u = Arcsen\, x \;\wedge\; dv = \dfrac{x}{\sqrt{1-x^2}}\, dx \Rightarrow du = \dfrac{1}{\sqrt{1-x^2}}\, dx \;\wedge\; v = -\sqrt{1-x^2}$

$$\int (Arcsen\, x)^2\, dx = x\,(Arcsen\, x)^2 - 2\left(-\sqrt{1-x^2}\, Arcsen\, x - \int \frac{-\sqrt{1-x^2}}{\sqrt{1-x^2}}\, dx\right)$$

$$\int (Arcsen\, x)^2\, dx = x\,(Arcsen\, x)^2 + 2\sqrt{1-x^2}\, Arcsen\, x + \int\, dx$$

$$\int (Arcsen\, x)^2\, dx = x\,(Arcsen\, x)^2 + 2\sqrt{1-x^2}\, Arcsen\, x + x + C$$

16. $\displaystyle\int e^{Arcsen\, x}\, dx$

Sea $u = e^{Arcsen\, x} \;\wedge\; dv = \; dx \Rightarrow du = \dfrac{e^{Arcsen\, x}}{\sqrt{1-x^2}}\, dx \;\wedge\; v = x$

$$\int e^{Arcsen\, x}\, dx = x\, e^{Arcsen\, x} - \int x\, \frac{e^{Arcsen\, x}}{\sqrt{1-x^2}}$$

Sea $u = e^{Arcsen\, x} \;\wedge\; dv = \dfrac{x}{\sqrt{1-x^2}}\, dx \Rightarrow du = \dfrac{e^{Arcsen\, x}}{\sqrt{1-x^2}}\, dx \;\wedge\; v = -\sqrt{1-x^2}$

$$\int e^{Arcsen\,x}\,dx = x\,e^{Arcsen\,x} - \left(-\sqrt{1-x^2}\,e^{Arcsen\,x} - \int -\sqrt{1-x^2}\frac{e^{Arcsen\,x}}{\sqrt{1-x^2}}\,dx\right)$$

$$\int e^{Arcsen\,x}\,dx = x\,e^{Arcsen\,x} + \sqrt{1-x^2}\,e^{Arcsen\,x} - \int e^{Arcsen\,x}\,dx$$

$$2\int e^{Arcsen\,x}\,dx = x\,e^{Arcsen\,x} + \sqrt{1-x^2}\,e^{Arcsen\,x}$$

$$\int e^{Arcsen\,x}\,dx = \frac{x\,e^{Arcsen\,x}}{2} + \frac{\sqrt{1-x^2}\,e^{Arcsen\,x}}{2}$$

17. $\displaystyle\int x^4\sqrt{1-x^2}\,dx = \int x^3\,x\sqrt{1-x^2}\,dx$

Sea $u = x^3 \;\wedge\; dv = x\sqrt{1-x^2}\,dx \Rightarrow du = 3x^2\,dx \;\wedge\; v = -\sqrt{1-x^2}$

$$\int x^4\sqrt{1-x^2}\,dx = -x^3\sqrt{1-x^2} + \int 3x^2\sqrt{1-x^2}\,dx$$

Sea $t^2 = 1-x^2 \Rightarrow 2t\,dt = -2x\,dx \wedge x^2 = 1-t^2$

$$\int x^4\sqrt{1-x^2}\,dx = -x^3\sqrt{1-x^2} - 3\int (1-t^2)\,t\,dt = -x^3\sqrt{1-x^2} - 3\int t - t^3\,dt$$

$$\int x^4\sqrt{1-x^2}\,dx = -x^3\sqrt{1-x^2} - 3\frac{t^2}{2} - 3\frac{t^4}{4} + C$$

$$\int x^4\sqrt{1-x^2}\,dx = -x^3\sqrt{1-x^2} - 3\frac{1-x^2}{2} - 3\frac{(1-x^2)^2}{4} + C$$

18. $\displaystyle\int x^2\,e^x\,sen\,x\,dx$

Sea $u = e^x\,x^2 \;\wedge\; dv = sen\,x\,dx \Rightarrow du = e^x\,x^2 + 2x\,e^x\,dx \;\wedge\; v = -cos\,x$

$$\int x^2\,e^x\,sen\,x\,dx = -e^x\,x^2\,cos\,x + \int cos\,x(e^x\,x^2 + 2x\,e^x)\,dx$$

$$\int x^2\,e^x\,sen\,x\,dx = -e^x\,x^2\,cos\,x + \int cos\,x\,e^x\,x^2\,dx + \int cos\,x\,2x\,e^x\,dx$$

$$\int x^2\,e^x\,sen\,x\,dx = -e^x\,x^2\,cos\,x + \int x^2\,e^x\,cos\,x\,dx + 2\int x\,e^x\,cos\,x\,dx$$

Sea $u = x^2\,e^x \;\wedge\; dv = cos\,x\,dx \Rightarrow du = 2x\,e^x + x^2\,e^x\,dx \;\wedge\; v = sen\,x$

$$\int x^2\,e^x\,cos\,x\,dx = x^2\,e^x\,sen\,x - \int sen\,x(2x\,e^x + x^2\,e^x)\,dx$$

$$\int x^2\,e^x\,cos\,x\,dx = x^2\,e^x\,sen\,x - \int 2x\,e^x\,sen\,x\,dx - \int x^2\,e^x\,sen\,x\,dx$$

$$2\int x^2\,e^x\,sen\,x\,dx = -e^x\,x^2\,cos\,x + x^2\,e^x\,sen\,x - 2\int x\,e^x\,sen\,x\,dx + 2\int x\,e^x\,cos\,x\,dx$$

Sea $u = x\,e^x \;\wedge\; dv = cos\,x\,dx \Rightarrow du = x\,e^x + e^x\,dx \;\wedge\; v = sen\,x$

$$\int x\,e^x \cos x\;dx = x\,e^x \operatorname{sen} x - \int \operatorname{sen} x(x\,e^x + e^x)\;dx$$

$$\int x\,e^x \cos x\;dx = x\,e^x \operatorname{sen} x - \int x\,e^x \operatorname{sen} x\;dx - \int \operatorname{sen} x\,e^x\;dx$$

$$2\int x^2\;e^x\;\operatorname{sen} x\;dx = -e^x\,x^2 \cos x + x^2\,e^x \operatorname{sen} x - 2\int x\,e^x \operatorname{sen} x\;dx$$

$$+\,2\left(x\,e^x \operatorname{sen} x - \int x\,e^x \operatorname{sen} x\;dx - \int \operatorname{sen} x\,e^x\;dx\right)$$

$$2\int x^2\;e^x\;\operatorname{sen} x\;dx = -e^x\,x^2 \cos x + x^2\,e^x \operatorname{sen} x - 2\int x\,e^x \operatorname{sen} x\;dx$$

$$+\,2x\,e^x \operatorname{sen} x - 2\int x\,e^x \operatorname{sen} x\;dx - 2\int \operatorname{sen} x\,e^x\;dx$$

$$2\int x^2\;e^x\;\operatorname{sen} x\;dx = -e^x\,x^2 \cos x + x^2\,e^x \operatorname{sen} x + 2x\,e^x \operatorname{sen} x -$$

$$4\int x\,e^x \operatorname{sen} x\;dx - 2\int \operatorname{sen} x\,e^x\;dx$$

Sea $u = x\,e^x \;\wedge\; dv = \operatorname{sen} x\;dx \Rightarrow du = x\,e^x + e^x\;dx \;\wedge\; v = -\cos x$

$$\int x\,e^x \operatorname{sen} x\;dx = -x\,e^x \cos x + \int \cos x(x\,e^x + e^x)\;dx$$

$$\int x\,e^x \operatorname{sen} x\;dx = -x\,e^x \cos x + \int x\,e^x \cos x\;dx + \int e^x \cos x\;dx$$

$$\int x\,e^x \operatorname{sen} x\;\;dx \;=\; -x\,e^x \cos x + \left(x\,e^x \operatorname{sen} x - \int x\,e^x \operatorname{sen} x\;dx - \int \operatorname{sen} x\,e^x\;dx\right) +$$

$$\int e^x \cos x\;dx$$

$$2\int x\,e^x \operatorname{sen} x\;dx = -x\,e^x \cos x + x\,e^x \operatorname{sen} x - \int \operatorname{sen} x\,e^x\;dx + \int e^x \cos x\;dx$$

$$2\int x^2\;e^x\;\operatorname{sen} x\;dx = -e^x\,x^2 \cos x + x^2\,e^x \operatorname{sen} x + 2x\,e^x \operatorname{sen} x -$$

$$2\left(-x\,e^x \cos x + x\,e^x \operatorname{sen} x - \int \operatorname{sen} x\,e^x\;dx + \int e^x \cos x\;dx\right) - 2\int \operatorname{sen} x\,e^x\;dx$$

$$2\int x^2\;e^x\;\operatorname{sen} x\;dx = -e^x\,x^2 \cos x + x^2\,e^x \operatorname{sen} x + 2x\,e^x \cos x - 2\int e^x \cos x\;dx$$

Sea $u = e^x \;\wedge\; dv = \cos x\;dx \Rightarrow du = e^x\;dx \;\wedge\; v = \operatorname{sen} x$

$$\int e^x \cos x\;dx = e^x \operatorname{sen} x - \int e^x \operatorname{sen} x$$

Sea $u = e^x \;\wedge\; dv = \operatorname{sen} x\;dx \Rightarrow du = e^x\;dx \;\wedge\; v = -\cos x$

$$\int e^x \cos x\;dx = e^x \operatorname{sen} x - \left(-e^x \cos x + \int e^x\,e^x\;dx\right)$$

$$2\int e^x\, cos\, x\; dx = e^x\, sen\, x + e^x\, cos\, x$$

$$2\int x^2\, e^x\; sen\, x\; dx = 2x\, e^x\, cos\, x - e^x\, x^2\, cos\, x + x^2\, e^x\, sen\, x - e^x\, sen\, x - e^x\, cos\, x$$

$$\int x^2\, e^x\; sen\, x\; dx = x\, e^x\, cos\, x + \frac{x^2\, e^x\, sen\, x - e^x\, x^2\, cos\, x - e^x\, sen\, x - e^x\, cos\, x}{2} + C$$

19. $\displaystyle\int \frac{x^2\; Arctg\, x}{1+x^2}\; dx$

Sea $t = Arctg\, x \Rightarrow dt = \dfrac{1}{1+x^2}\; dx \;\wedge\; tg\, t = x$

$$\int \frac{x^2\; Arctg\, x}{1+x^2}\; dx = \int t\, tg^2\, t\; dt = \int t\,(sec^2\, t - 1)\; dt = \int t\, sec^2\, t\; dt - \int t\; dt$$

Sea $u = t \;\wedge\; dv = sec^2\, t\; dt \Rightarrow du = \; dt \;\wedge\; v = tg\, t$

$$\int t\, sec^2\, t\; dt - \int t\; dt = t\, tg\, t - \int tg\, t\; dt - \frac{t^2}{2} + C$$

$$\int t\, sec^2\, t\; dt - \int t\; dt = t\, tg\, t + ln|cos\, t| - \frac{t^2}{2} + C$$

$$\int \frac{x^2\; Arctg\, x}{1+x^2}\; dx = Arctg\, x\, tg\, Arctg\, x + ln|\frac{1}{\sqrt{1+x^2}}| - \frac{Arctg^2\, x}{2} + C$$

$$\int \frac{x^2\; Arctg\, x}{1+x^2}\; dx = x\, Arctg\, x + ln|\frac{1}{\sqrt{1+x^2}}| - \frac{Arctg^2\, x}{2} + C$$

Integración de formas cuadráticas y fracciones parciales

20. $\displaystyle\int \frac{1}{\sqrt{4x-3-x^2}}\; dx = \int \frac{1}{\sqrt{-(x^2-4x+3)}}\; dx = \int \frac{1}{\sqrt{1-(x^2-4x+4)}}\; dx =$

$$\int \frac{1}{\sqrt{1-(x-2)^2}}\; dx = Arcsen\,(x-2) + C$$

21. $\displaystyle\int \frac{2x+1}{2x^2+x+1}\; dx = \frac{1}{2}\int \frac{4x+2}{2x^2+x+1}\; dx = \frac{1}{2}\int \frac{4x+1}{2x^2+x+1} + \frac{1}{2x^2+x+1}\; dx =$

$$\frac{1}{2}\int \frac{4x+1}{2x^2+x+1}\; dx + \frac{1}{2}\int \frac{1}{2x^2+x+1}\; dx = \frac{1}{2}\, ln\,|2x^2+x+1| + \frac{1}{2}\int \frac{1}{2x^2+x+1}\; dx =$$

$$\frac{1}{2}\, ln\,|2x^2+x+1| + \frac{1}{2}\int \frac{1}{2\left(x+\frac{1}{4}\right)^2+\frac{7}{8}}\; dx = \frac{1}{2}\, ln\,|2x^2+x+1| + \frac{1}{4}\int \frac{1}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}\; dx =$$

$$\frac{1}{2}\, ln\,|2x^2+x+1| + \frac{1}{4}\frac{\sqrt{7}}{4} Arctg\, \frac{4(x+\frac{1}{4})}{\sqrt{7}} + C = \frac{1}{2}\, ln\,|2x^2+x+1| + \frac{\sqrt{7}}{16} Arctg\, \frac{4x+1}{\sqrt{7}} + C$$

22. $\displaystyle\int \frac{e^x}{e^{2x}+e^x+1}\; dx$

Sea $u = e^x \Rightarrow du = e^x\ dx$

$$\int \frac{e^x}{e^{2x}+e^x+1}\ dx = \int \frac{1}{u^2+u+1}\ du = \int \frac{1}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}\ du =$$

$$\frac{\sqrt{3}}{2} Arctg\left(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C = \frac{\sqrt{3}}{2} Arctg\left(\frac{2u+1}{\sqrt{3}}\right) + C = \frac{\sqrt{3}}{2} Arctg\left(\frac{2e^x+1}{\sqrt{3}}\right) + C$$

23. $$\int \frac{1}{x^6-1}\ dx = \int \frac{1}{(x^3)^2-1}\ dx = \int \frac{1}{(x^3-1)(x^3+1)}\ dx =$$

$$\int \frac{1}{(x-1)(x^2+x+1)(x+1)(x^2-x+1)}\ dx =$$

$$\int \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} + \frac{D}{x+1} + \frac{Ex+F}{x^2-x+1}\ dx =$$

$$\frac{1}{6}\int \frac{1}{x-1} - \frac{x+2}{x^2+x+1} - \frac{1}{x+1} + \frac{x-2}{x^2-x+1}\ dx =$$

$$\frac{1}{6} ln\,|x-1| - \frac{1}{6} ln\,|x+1| - \frac{1}{6}\int \frac{x+2}{x^2+x+1}\ dx + \frac{1}{6}\int \frac{x-2}{x^2-x+1}\ dx =$$

$$\frac{1}{6} ln\,|x-1| - \frac{1}{6} ln\,|x+1| - \frac{1}{12}\int \frac{2x+4}{x^2+x+1}\ dx + \frac{1}{12}\int \frac{2x-4}{x^2-x+1}\ dx =$$

$$\frac{1}{6} ln\left|\frac{x-1}{x+1}\right| - \frac{1}{12}\int \frac{2x+1}{x^2+x+1} + \frac{3}{x^2+x+1}\ dx + \frac{1}{12}\int \frac{2x-1}{x^2-x+1} - \frac{3}{x^2-x+1}\ dx =$$

$$\frac{1}{6}\ ln\left|\frac{x-1}{x+1}\right| - \frac{1}{12}\ ln\,|x^2+x+1| + \frac{1}{12}\ ln\,|x^2-x+1| + \frac{1}{12}\int \frac{3}{x^2+x+1}\ dx - \frac{1}{12}\int \frac{3}{x^2-x+1}\ dx =$$

$$\frac{1}{6}\ ln\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\ ln\left|\frac{x^2-x+1}{x^2+x+1}\right| + \frac{1}{12}\int \frac{3}{(x+1)^2+\frac{3}{4}}\ dx - \frac{1}{12}\int \frac{3}{(x-1)^2+\frac{3}{4}}\ dx =$$

$$\frac{1}{6}\ ln\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\ ln\left|\frac{x^2-x+1}{x^2+x+1}\right| + \frac{\sqrt{3}}{24}\ Arctg\left(\frac{2x+2}{\sqrt{3}}\right) - \frac{\sqrt{3}}{24}\ Arctg\left(\frac{2x-2}{\sqrt{3}}\right) + C$$

24. $$\int \frac{1}{x^4+4}\ dx = \int \frac{1}{(x^2+2x+2)(x^2-2x+2)}\ dx =$$

$$\int \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{x^2-2x+2}\ dx$$

25. $\displaystyle\int \frac{3x-1}{x^3+x^2+x+1}\,dx = \int \frac{3x-1}{(x+1)(x^2+1)}\,dx = \int \frac{A}{x+1} + \frac{Bx+C}{x^2+1}\,dx$

26. $\displaystyle\int \frac{1}{(x^2-x)(x^2-x+1)^2}\,dx = \int \frac{1}{x(x-1)(x^2-x+1)^2}\,dx =$

$\displaystyle\int \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2-x+1} + \frac{Ex+F}{(x^2-x+1)^2}\,dx$

27. $\displaystyle\int \sqrt{tg\,x}\,dx = \int \frac{\sqrt{tg\,x}\,sec^2\,x}{sec^2\,x}\,dx = \int \frac{\sqrt{tg\,x}\,sec^2\,x}{x^2+1}\,dx$

Sea $u^2 = tg\,x \Rightarrow 2u\,du = sec^2\,x\,dx$

$\displaystyle\int \frac{\sqrt{tg\,x}\,sec^2\,x}{x^2+1}\,dx = \int \frac{\sqrt{u^2}\,2u}{u^4+1}\,du = 2\int \frac{u^2}{u^4+1}\,du =$

$\displaystyle\frac{x^2}{1+x^4} = \frac{x^2}{(x^2+\sqrt{2}x+1)(x^2-x\sqrt{2}+1)} =$

$\displaystyle\frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$ de donde:

$x^2 = (A+C)x^3 + (-\sqrt{2}A + B + \sqrt{2}C + D)x^2 + (A - \sqrt{2}B + C + \sqrt{2}D)x + B + D$

$A + C = 0$

$-\sqrt{2}A + B + \sqrt{2}C + D = 1$

$A - \sqrt{2}B + C + \sqrt{2}D = 0$

$B + D = 0$ resolviendo el sistema encontramos:

$A = -\dfrac{1}{2\sqrt{2}} \quad B = D = 0$ y $C = \dfrac{1}{2\sqrt{2}}$ entonces:

$$\int \frac{x^2}{1+x^4}\,dx = -\frac{1}{2\sqrt{2}}\int \frac{x}{x^2+\sqrt{2}x+1}\,dx + \frac{1}{2\sqrt{2}}\int \frac{x}{x^2-\sqrt{2}x+1}\,dx =$$

$$-\frac{1}{4\sqrt{2}}\int \frac{(2x+\sqrt{2})}{x^2+\sqrt{2}x+1}\,dx + \frac{1}{4}\int \frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\,dx +$$

$$\frac{1}{4\sqrt{2}}\int \frac{(2x-\sqrt{2})}{x^2-\sqrt{2}x+1}\,dx + \frac{1}{4}\int \frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\,dx$$

$$= -\frac{1}{4\sqrt{2}}\,ln\,(x^2+\sqrt{2}x+1) + \frac{\sqrt{2}}{4}Arctg\,\frac{x+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} +$$

$$\frac{1}{4\sqrt{2}}\,ln\,(x^2-\sqrt{2}x+1) + \frac{\sqrt{2}}{4}Arctg\,\frac{x-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} + C =$$

$$\frac{1}{4\sqrt{2}}\left[ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + 2Arctg\,(1+\sqrt{2}x) + 2\,Arctg\,(\sqrt{2}x-1)\right] + C$$

$$\int \frac{u^2}{u^4+1}\,du = 2\frac{1}{4\sqrt{2}}\left[ln\left|\frac{u^2-u\sqrt{2}+1}{u^2+u\sqrt{2}+1}\right| + 2Arctg\,(1+u\sqrt{2}) + 2\,Arctg\,(u\sqrt{2}-1)\right] + C =$$

$$\frac{1}{2\sqrt{2}}\left[ln\left|\frac{u^2-u\sqrt{2}+1}{u^2+u\sqrt{2}+1}\right| + 2Arctg\,(1+u\sqrt{2}) + 2\,Arctg\,(u\sqrt{2}-1)\right] + C =$$

$$\frac{1}{2\sqrt{2}}\left[ln\left|\frac{tg\,x-\sqrt{2tg\,x}+1}{tg\,x+\sqrt{2tg\,x}+1}\right| + 2Arctg\,(1+\sqrt{2tg\,x}) + 2\,Arctg\,(\sqrt{2tg\,x}-1)\right] + C$$

$$\int \sqrt{tg\,x}\,dx = \frac{1}{2\sqrt{2}}\left[ln\left|\frac{tg\,x-\sqrt{2tg\,x}+1}{tg\,x+\sqrt{2tg\,x}+1}\right| + 2Arctg\,(1+\sqrt{2tg\,x}) + 2\,Arctg\,(\sqrt{2tg\,x}-1)\right] + C$$

28. Demuestre que

$$\lim_{t\to\infty}\int_0^t \frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}\,dx = \frac{\pi}{2(a+b)(b+c)(c+a)}$$

$$\frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+b^2} + \frac{Ex+F}{x^2+c^2} \Rightarrow$$

$$(Ax+B)(x^2+b^2)(x^2+c^2)+(Cx+D)(x^2+a^2)(x^2+c^2)+(Ex+F)(x^2+a^2)(x^2+b^2) = x^2$$

Resolviendo

$A + C + E = 0$

$B + D + F = 0$

$Ac^2 + Ea^2 + Cc^2 + b^2E + b^2A + Ca^2 = 0$

$Da^2 + Dc^2 + Bc^2 + b^2F + Fa^2 + b^2B = 1$

$Ab^2c^2 + Ea^2b^2 + Ca^2c^2 = 0$

$Da^2c^2 + Bb^2c^2 + Fa^2b^2 = 0 \Rightarrow$

$A = 0, E = 0, C = 0, F = \dfrac{-c^2}{(a^2-c^2)(b^2-c^2)}, B = \dfrac{-a^2}{(a^2-c^2)(a^2-b^2)},$

$D = \dfrac{b^2}{(b^2-c^2)(a^2-b^2)}$

$\dfrac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \dfrac{-a^2}{(a^2-c^2)(a^2-b^2)}\dfrac{1}{x^2+a^2} + \dfrac{b^2}{(b^2-c^2)(a^2-b^2)}\dfrac{1}{x^2+b^2} +$

$\dfrac{-c^2}{(a^2-c^2)(b^2-c^2)}\dfrac{1}{x^2+c^2}$

$\displaystyle\int \frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}\,dx = \frac{-a^2}{(a^2-c^2)(a^2-b^2)}\int\frac{1}{x^2+a^2}\,dx +$

$\displaystyle\frac{b^2}{(b^2-c^2)(a^2-b^2)}\int\frac{1}{x^2+b^2}\,dx + \frac{-c^2}{(a^2-c^2)(b^2-c^2)}\int\frac{1}{x^2+c^2}\,dx =$

$\dfrac{-a^2}{(a^2-c^2)(a^2-b^2)}\dfrac{1}{a}\,Arctg\left(\dfrac{x}{a}\right) + \dfrac{b^2}{(b^2-c^2)(a^2-b^2)}\dfrac{1}{b}\,Arctg\left(\dfrac{x}{b}\right) +$

$\dfrac{-c^2}{(a^2-c^2)(b^2-c^2)}\dfrac{1}{c}\,Arctg\left(\dfrac{x}{c}\right) =$

$\dfrac{-a}{(a^2-c^2)(a^2-b^2)}\,Arctg\left(\dfrac{x}{a}\right) + \dfrac{b}{(b^2-c^2)(a^2-b^2)}\,Arctg\left(\dfrac{x}{b}\right) - \dfrac{c}{(a^2-c^2)(b^2-c^2)}\,Arctg\left(\dfrac{x}{c}\right)$

$\displaystyle\int_0^t \frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}\,dx = \frac{-a}{(a^2-c^2)(a^2-b^2)}\,Arctg\left(\frac{t}{a}\right) +$

$\dfrac{b}{(b^2-c^2)(a^2-b^2)}\,Arctg\left(\dfrac{t}{b}\right) - \dfrac{c}{(a^2-c^2)(b^2-c^2)}\,Arctg\left(\dfrac{t}{c}\right) +$

$\dfrac{a}{(a^2-c^2)(a^2-b^2)}\,Arctg\left(\dfrac{0}{a}\right) - \dfrac{b}{(b^2-c^2)(a^2-b^2)}\,Arctg\left(\dfrac{0}{b}\right) +$

$\dfrac{c}{(a^2-c^2)(b^2-c^2)}\,Arctg\left(\dfrac{0}{c}\right)$

$\displaystyle\int_0^t \frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}\,dx = \frac{-a}{(a^2-c^2)(a^2-b^2)}\,Arctg\left(\frac{t}{a}\right) +$

$\dfrac{b}{(b^2-c^2)(a^2-b^2)}\,Arctg\left(\dfrac{t}{b}\right) - \dfrac{c}{(a^2-c^2)(b^2-c^2)}\,Arctg\left(\dfrac{t}{c}\right)$

$\displaystyle\lim_{t\to\infty}\int_0^t \frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}\,dx = \lim_{t\to\infty}\frac{-a}{(a^2-c^2)(a^2-b^2)}\,Arctg\left(\frac{t}{a}\right) +$

$\dfrac{b}{(b^2-c^2)(a^2-b^2)}\,Arctg\left(\dfrac{t}{b}\right) - \dfrac{c}{(a^2-c^2)(b^2-c^2)}\,Arctg\left(\dfrac{t}{c}\right) =$

$$\frac{-a}{(a^2-c^2)(a^2-b^2)}\,\frac{\pi}{2}+\frac{b}{(b^2-c^2)(a^2-b^2)}\,\frac{\pi}{2}-\frac{c}{(a^2-c^2)(b^2-c^2)}\,\frac{\pi}{2}=$$

$$\frac{\pi}{2}\left(\frac{-a}{(a^2-c^2)(a^2-b^2)}+\frac{b}{(b^2-c^2)(a^2-b^2)}-\frac{c}{(a^2-c^2)(b^2-c^2)}\right)=$$

$$\frac{\pi}{2}\left(\frac{-a(b^2-c^2)+b(a^2-c^2)-c(a^2-b^2)}{(a^2-c^2)(a^2-b^2)(b^2-c^2)}\right)=$$

$$\frac{\pi}{2}\left(\frac{(a-b)(b-c)(a-c)}{(a-c)(a+c)(a-b)(a+b)(b-c)(b+c)}\right)=$$

$$\frac{\pi}{2}\left(\frac{1}{(a+c)(a+b)(b+c)}\right)=\frac{\pi}{2(a+c)(a+b)(b+c)}$$